Constitutive Relations for Moving Plasmas

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A covariant form of Ohm's Law for bianisotropic plasmas is set up connecting the four-dimensional current density with the field tensor through a material tensor of order three. This tensor is represented by two four-dimensional material tensors of order two, which are closely related to the usual threedimensional conductivity tensors; its symmetry properties are investigated and relations between its components and those of the three-dimensional material tensors are established. In addition a covariant constitutive equation for a plasma is formulated using the polarization model, where the four-dimensional current density is substituted by a polarization tensor. Thereby the plasma properties — like the dielectric and magnetic properties of a medium — are expressed by a material tensor of order four, whose representation is generalized for bianisotropic media.

1. Introduction

The material properties of moving plasmas are sometimes described by four-dimensional covariant constitutive equations, which are independent of the observer, or by the more common three-dimensional forms. The latter can be obtained — after some manipulations — from the three dimensional constitutive equations in the comoving frame by substituting the field components through their Lorentz-transformed counterparts in the observer's frame [1].

Covariant constitutive equations have been investigated in the literature mainly for bound charges, where the dielectric and magnetic properties of the medium are combined by a material tensor of order four, connecting the field tensor with the excitation tensor. Marx [2] and Schmutzer [3] have formulated such covariant constitutive relations for anisotropic media. They represented the material tensor of order four by tensors of order two, which are closely related to the three-dimensional electric and magnetic permittivity tensors in the comoving frame. Kong [4] dealt with bianisotropic media and established relations between the components of the fourth-order material tensor and the components of the three-dimensional material tensors.

Analogous to the covariant constitutive relation for bound charges we will formulate a covariant constitutive relation for free charges (a covariant Ohm's Law), where now the four-dimensional conduction current is connected with the field tensor by a material tensor of order three. Instead of the conduction current it is as well possible to connect the total current (conduction and convection current) with the field tensor. This leads, however, to a somewhat different form of Ohm's Law with a different material tensor of order three. Both material tensors will be investigated in detail. Their symmetry properties will be studied and, as in the case of bound charges, their relations to the three-dimensional conductivity tensor in the comoving frame as well as in the observer's frame. The covariant Ohm's Law and the associated material tensor include the special forms, which, for example, have been given by Pauli [5] for an isotropic medium and by Schmutzer [6] for an anisotropic medium using a material tensor of order two.

The response of the free charges (conductivity tensor) is sometimes formally expressed by the response of bound charges (electric and magnetic susceptibility tensors) in the so-called polarization model, using the relation [5] between the polarization tensor and the four-dimensional current. A covariant constitutive equation for free charges in the polarization model was formulated by Derfler and O'Sullivan [7], who calculated the components of their susceptibility tensor of order four — connecting the polarization tensor with the field tensor — microscopically by means of a covariant Vlasov equation. This susceptibility tensor possesses the same structure as the material tensor of order four for bound charges. Therefore the results found for the latter can be adopted and, as far as it is necessary, they will be generalized for bianisotropic media.

Another covariant formulation of the constitutive equation for free charges, by Tischer and Hess [8],

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should be mentioned here without further considerations. Instead of the four-dimensional current vector the authors introduced a skew-symmetric current tensor, consisting of an electric and magnetic current, where the magnetic current is solely caused by the convection.

2. Ohm's Law in Moving Media

a) Covariant Formulation

The covariant Ohm's Law is given by the connection of the four-dimensional conduction current $(\delta^\nu_\mu + u^\nu u_\mu)\, j^\mu$ with the field tensor $F^{\lambda\varkappa}$ by a tensor of order three in the form

$$(\delta^\nu_\mu + u^\nu u_\mu)\, j^\mu = \tfrac{1}{2}\, \sigma^\nu{}_{\lambda\varkappa}\, F^{\lambda\varkappa} \tag{2.1}$$

with $j^\mu := (\varrho c, j^1, j^2, j^3)$ the four-dimensional current density vector and the field tensor $F^{\lambda\varkappa}$, whose components are defined by

$$F^{0l} := E^l, \quad F^{lk} := \varepsilon^{lkm}\, c\, B_m\,.$$

Furthermore $u^\nu := (c, v^1, v^2, v^3)/\sqrt{c^2 - v^2}$ is the normalized four-velocity ($u^\nu u_\nu = -1$) with v^l as mean center of mass velocity of the whole medium in the observer's frame.

Throughout the paper SI units are used, greek indices are running from 0 to 3, latin indices from 1 to 3. The material tensor $\sigma^\nu{}_{\lambda\varkappa}$ combines the conduction properties of the medium and may depend in general on differential and integral operators (for time and space dispersive media and for inhomogeneous media) and on the field components themselves for nonlinear media. The factor $\frac{1}{2}$ has been introduced for convenience. The skew-symmetry of $F^{\lambda\varkappa}$ and Eq. (2.1) yield

$$\sigma^\nu{}_{\lambda\varkappa} = -\,\sigma^\nu{}_{\varkappa\lambda}\,, \quad u_\nu\, \sigma^\nu{}_{\lambda\varkappa} = 0\,, \tag{2.2}$$

which reduce the originally 64 components to 18.

For analytical calculations it is more convenient to represent $\sigma^\nu{}_{\lambda\varkappa}$ by material tensors of order two. To do this we first introduce four-fields E^λ, B^λ defined by

$$E^\lambda := F^{\lambda\varkappa} u_\varkappa, \quad c\, B^\lambda := \mathscr{F}^{\lambda\varkappa} u_\varkappa,$$

$$E^\lambda = (E^l v_l/c, E^l + \varepsilon^{lkm} v_k B_m)/(1 - v^2/c^2)^{1/2}, \tag{2.3a}$$

$$B^\lambda = (B^l v_l/c, B^l - \varepsilon^{lkm} v_k E_m/c^2)/(1 - v^2/c^2)^{1/2}$$

with the dual field tensor

$$\mathscr{F}^{\lambda\varkappa} := \tfrac{1}{2}\, \varepsilon^{\lambda\varkappa\varphi\psi} F_{\varphi\psi}\,,$$

$$\mathscr{F}^{0k} = -\,c\, B^k, \quad \mathscr{F}^{lk} = -\,\varepsilon^{lkm} E_m$$

where $\varepsilon^{\lambda\varkappa\varphi\psi}$ is the Levi-Civita symbol. In the comoving frame the space components of the four-fields E^λ, B^λ are the three-dimensional fields E'^l and B'^l, while the time components vanish. These four-fields E^λ, B^λ are perpendicular to the four-velocity,

$$E^\lambda u_\lambda = 0\,, \quad B^\lambda u_\lambda = 0\,, \tag{2.3b}$$

which follows from the skew-symmetry of the field tensors $F^{\lambda\varkappa}$, $\mathscr{F}^{\lambda\varkappa}$ and the definition of the four-fields (2.3a). The field tensor $F^{\lambda\varkappa}$ can be represented by these four-fields as [2]

$$F^{\lambda\varkappa} = u^\lambda E^\varkappa - u^\varkappa E^\lambda - \varepsilon^{\lambda\varkappa\varphi\psi} u_\varphi\, c\, B_\psi\,. \tag{2.4}$$

With the help of these four-fields one can formulate Ohm's Law as following:

$$(\delta^\nu_\mu + u^\nu u_\mu)\, j^\mu = {}_E\sigma^\nu{}_\lambda\, E^\lambda + {}_B\sigma^\nu{}_\varphi\, c\, B^\varphi\,, \tag{2.5}$$

which together with [c. f. (2.3a)]

$$\begin{aligned}(\delta^\nu_\mu &+ u^\nu u_\mu)\, j^\mu \\ &= \tfrac{1}{2}\,({}_E\sigma^\nu{}_\lambda\, E^\lambda + {}_E\sigma^\nu{}_\varkappa\, E^\varkappa + 2\, {}_B\sigma^\nu{}_\varphi\, c\, B^\varphi) \qquad (2.6)\\ &= \tfrac{1}{2}\,({}_E\sigma^\nu{}_\lambda\, u_\varkappa - {}_E\sigma^\nu{}_\varkappa\, u_\lambda + {}_B\sigma^\nu{}_\varphi\, \varepsilon_{\lambda\varkappa}{}^{\varphi\psi} u_\psi)\, F^{\lambda\varkappa}\end{aligned}$$

leads by comparison with (2.1) to the desired representation of $\sigma^\nu{}_{\lambda\varkappa}$ by the material tensors of order two

$$\sigma^\nu{}_{\lambda\varkappa} = {}_E\sigma^\nu{}_\lambda\, u_\varkappa - {}_E\sigma^\nu{}_\varkappa\, u_\lambda + {}_B\sigma^\nu{}_\varphi\, \varepsilon_{\lambda\varkappa}{}^{\varphi\psi} u_\psi\,. \tag{2.7}$$

In the case of an isotropic medium Ohm's Law (2.1) with (2.7) becomes

$$\begin{aligned}(\delta^\nu_\mu &+ u^\nu u_\mu)\, j^\mu \\ &= {}_E\sigma\, u_\varkappa\, F^{\nu\varkappa} + \tfrac{1}{2}\, {}_B\sigma\, \varepsilon_{\lambda\varkappa}{}^{\nu\psi} u_\psi\, F^{\lambda\varkappa}\,.\end{aligned} \tag{2.8}$$

For ${}_B\sigma = 0$ this is Ohm's Law as formulated by Pauli [5].

As the conduction current and the four-fields E^λ, B^φ are perpendicular to the four-velocity, which implies that they have only three linearly independent components, only 9 components of the tensors ${}_{E,B}\sigma^\nu{}_\lambda$ are independent. Contraction of (2.5) with u_ν yields with (2.3b) and arbitrary ${}_{E,B}\bar\sigma$

$$u_\nu\, {}_{E,B}\sigma^\nu{}_\lambda = {}_{E,B}\bar\sigma\, u_\lambda \tag{2.9a}$$

as a sufficient condition for the vanishing of the contracted right hand side of (2.5). Because of the orthogonality relation (2.3b) ${}_{E,B}\sigma^\nu{}_\lambda\, u^\lambda$ can take arbitrary values. In order to maintain possible symmetries we choose

$${}_{E,B}\sigma^\nu{}_\lambda\, u^\lambda = {}_{E,B}\bar\sigma\, u^\nu\,. \tag{2.9b}$$

In order that in the comoving frame Eq. (2.5) for $\nu = l$ becomes the usual Ohm's Law, viz.

$$j'^l = {}_E\sigma'^l{}_k\, E'^k + {}_B\sigma'^l{}_k\, c\, B'^k \tag{2.10}$$

the pure space components of ${}_{E,B}\sigma'^\nu{}_\lambda$ have to be equal to the components of the three-dimensional conductivity tensor. Setting $\nu = 0$ in Eq. (2.5) leads to the identity $0 = 0$. Separation of the space and time components in condition (2.9a) and assumption (2.9b) determines in the comoving frame the remaining components of ${}_{E,B}\sigma'^\nu{}_\lambda$. Thus in the comoving frame the material tensors ${}_{E,B}\sigma'^\nu{}_\lambda$ are given explicitly by

$$\begin{aligned} &{}_{E,B}\sigma'^l{}_k = {}_{E,B}\sigma'^l{}_k\,, \quad {}_{E,B}\sigma'^l{}_0 = 0 = {}_{E,B}\sigma'^0{}_k\,,\\ &{}_{E,B}\sigma'^0{}_0 = {}_{E,B}\overline{\sigma}\,, \end{aligned} \tag{2.11}$$

where the mixed space time components vanish, only the pure space components are determined and the pure time component is arbitrary. This arbitrary component may be but equal to zero (accordingly to [2]) or in order to avoid the singularity of ${}_{E,B}\sigma^\nu{}_\lambda$ (e.g. for inversion problems) chosen arbitrarily unequal to zero.

The representation of ${}_{E,B}\sigma^\nu{}_\lambda$ in the observer's frame is derived from (2.11) by Lorentz transformation in the form

$${}_{E,B}\sigma^\nu{}_\lambda = Q^\nu_\alpha\, {}_{E,B}\sigma'^\alpha{}_\beta\, P^\beta_\lambda \tag{2.12}$$

with the Lorentz transformations Q^ν_α, P^β_λ given by [4]

$$\begin{aligned} Q^l_k &= P^l_k = \Gamma^l_k\,, & Q^0_0 &= P^0_0 = \gamma\,, \\ Q^l_0 &= -\,P^l_0 = \gamma\, v^l/c\,, & Q^0_k &= -\,P^0_k = \gamma\, v_k/c\,, \end{aligned} \tag{2.13a}$$

and

$$\begin{aligned} \Gamma^l_k &:= \delta^l_k + (\gamma - 1)\, v_k\, v^l/v^2\,,\\ \gamma &:= (1 - v^2/c^2)^{-1/2}\,. \end{aligned} \tag{2.13b}$$

The material tensors of order two ${}_E\sigma'^\nu{}_\lambda({}_B\sigma'^\nu{}_\lambda)$ can be identified in the comoving frame with the response of the free charges on the electric field (magnetic flux). In the observer's frame such an identification is not possible, since the four-fields E^λ, B^λ (2.3a) depend in general on both, the electric field E_l and the magnetic flux B_l.

Besides of the symmetry relation (2.2) of the material tensor due to the skew-symmetry of the field tensor $F^{\lambda\varkappa}$, one often requires additional symmetry relations. This was done, for example, by Post [9] for the material tensor for bound charges, in order to transfer symmetry properties of the three-dimensional material tensors of order two to the four-dimensional material tensor of order four.

According to Post [9] we investigate the vanishing of

$$\begin{aligned} \sigma^{[\nu\lambda\varkappa]} &:= \frac{1}{3!} \\ &\cdot (\sigma^{\nu\lambda\varkappa} + \sigma^{\varkappa\nu\lambda} + \sigma^{\lambda\varkappa\nu} - \sigma^{\nu\varkappa\lambda} - \sigma^{\lambda\nu\varkappa} - \sigma^{\varkappa\lambda\nu}) = 0\,, \end{aligned} \tag{2.14a}$$

which reduces the 18 components of $\sigma^{\nu\lambda\varkappa}$ to 14, and we look what condition this would impose on ${}_{E,B}\sigma^{\nu\lambda}$. The skew-symmetry (2.2) reduces (2.14a) to

$$\sigma^{\nu\lambda\varkappa} + \sigma^{\varkappa\nu\lambda} + \sigma^{\lambda\varkappa\nu} = 0\,. \tag{2.14b}$$

Inserting (2.7) into the above equation and rearranging lead to

$$\begin{aligned} &({}_E\sigma^{\nu\lambda} - {}_E\sigma^{\lambda\nu})\, u^\varkappa + ({}_E\sigma^{\varkappa\nu} - {}_E\sigma^{\nu\varkappa})\, u^\lambda \\ &\quad + ({}_E\sigma^{\lambda\varkappa} - {}_E\sigma^{\varkappa\lambda})\, u^\nu + {}_B\sigma^\nu{}_\varphi\, \varepsilon^{\lambda\varkappa\varphi\psi}\, u_\psi \\ &\quad + {}_B\sigma^\varkappa{}_\varphi\, \varepsilon^{\nu\lambda\varphi\psi}\, u_\psi + {}_B\sigma^\lambda{}_\varphi\, \varepsilon^{\varkappa\nu\varphi\psi}\, u_\psi = 0 \end{aligned} \tag{2.15}$$

from which one recognizes immediately that the symmetry of ${}_E\sigma^{\nu\lambda}$, viz.

$${}_E\sigma^{\nu\lambda} = {}_E\sigma^{\lambda\nu} \tag{2.16a}$$

is sufficient for the vanishing of the first six terms in (2.15). It is more convenient to consider the last three terms of (2.15) in the comoving frame, where one gets with $u'^\lambda = (1, 0, 0, 0)$

$${}_B\sigma'^n{}_p\, \varepsilon^{lkp} + {}_B\sigma'^k{}_p\, \varepsilon^{nlp} + {}_B\sigma'^l{}_p\, \varepsilon^{knp} = 0\,.$$

Because of the skew-symmetry of the Levi-Civita symbol this sum vanishes for all values of n, l, k except $n \neq l \neq k$, which requires the three-dimensional trace ${}_B\sigma'^l{}_l$ to be zero. As the four-dimensional trace ${}_B\sigma^\nu{}_\nu$ is Lorentz invariant, (2.11) yields

$${}_B\sigma^\nu{}_\nu = {}_B\overline{\sigma} \tag{2.16b}$$

as the condition on ${}_B\sigma^\nu{}_\lambda$ for the validity of (2.14a).

Because of (2.16a) the assumed symmetry relation (2.14) would require in an anisotropic medium the symmetry of ${}_E\sigma'^{lk}$. The conductivity tensor $\sigma^{\nu\lambda\varkappa}$ would be completely determined by the six components of ${}_E\sigma'^{lk}$. For bianisotropic media it would impose the additional condition (2.16b), namely the vanishing of the trace ${}_B\sigma'^l{}_l$. Both conditions (2.16) would restrict the media described by ${}_E\sigma^{\nu\lambda}$, ${}_B\sigma^{\nu\lambda}$ in an unwanted way, since many anisotropic media must be described by a non-symmetric ${}_E\sigma'^{lk}$ in the comoving frame. The condition (2.14) will therefore not be imposed in what follows.

b) Three-Dimensional Formulation

If we want to maintain the three-dimensional formulation, Ohm's Law must be given by the connection of the three-dimensional conduction current (total current j^k reduced by the convection current ϱv^k) with the field components in the form

$$j^k - \varrho\, v^k = {}_E\sigma^{*k}{}_n\, E^n + {}_B\sigma^{*k}{}_s\, c\, B^s\,, \qquad (2.17)$$

where the asterisks distinguish between the three-dimensional tensors in (2.17) and the space components of the four-dimensional ones, which coincide only in the comoving frame. The quantities ${}_{E,B}\sigma^{*l}{}_k$, unlike ${}_{E,B}\sigma^{\nu}{}_{\lambda}$ characterize the response of free charges on the electric field and magnetic flux, respectively, in the comoving frame as well as in the observer's frame.

Inserting in (2.17) the relations between the field components and the field tensor $F^{\lambda\varkappa}$ [4]

$$E^n = \tfrac{1}{2}\,(F^{0n} - F^{n0})\,,\quad c\,B^s = \tfrac{1}{2}\,\varepsilon^{smn}\,F_{mn} \qquad (2.18)$$

one gets by comparison of (2.17) with (2.1) for $\nu = l$ and splitting the summation over λ, $\varkappa$ into 0, l the following relations between $\sigma^{\nu\lambda\varkappa}$ and ${}_{E,B}\sigma^{*l}{}_k$

$$\begin{aligned}\sigma^{l0n} &= -\,\Gamma^{2\,l}_{\;k}\;{}_E\sigma^{*kn}\,,\\ \sigma^{lmn} &= \Gamma^{2\,l}_{\;k}\;{}_B\sigma^{*k}{}_s\;\varepsilon^{smn}\end{aligned} \qquad (2.19)$$

with

$$\Gamma^{2\,l}_{\;k} = \delta^l_k + (\gamma^2/c^2)\, v^l\, v_k\,.$$

The representation of $\sigma^{\nu\lambda\varkappa}$ by the components of the three-dimensional tensors ${}_{E,B}\sigma'^{l}{}_k$ in the comoving frame is given by the Lorentz transformation, according to (2.13), of Eq. (2.19) taken in the comoving frame. Comparison of these results with (2.19), as given in the observer's frame, leads after some manipulations to the relations

$$\begin{aligned}{}_E\sigma^{*kn} &= \gamma\, V^k_r\;{}_E\sigma'^{rs}\; V^n_s + \gamma\, V^k_r\;{}_B\sigma'^r{}_s\;\varepsilon^{nqs}\, v_q/c\,,\\ {}_B\sigma^{*k}{}_t &= \gamma\, V^k_r\;{}_E\sigma'^{rm}\;\varepsilon_{tmn}\, v^n/c + \gamma\, V^k_r\;{}_B\sigma'^r{}_s\; V^s_t\end{aligned} \qquad (2.20)$$

with

$$V^k_r := \Gamma^{-1k}{}_r = \delta^k_r - (1 - 1/\gamma)\, v^k\, v_r/v^2\,.$$

Expressions (2.20) represent the transformation relations of the three-dimensional material tensors. For anisotropic media (${}_B\sigma'^r{}_s = 0$) the last terms vanish. For an isotropic medium (2.20) reduces to

$$\begin{aligned}{}_E\sigma^{*kn} &= \gamma\,{}_E\sigma'\; V^{2kn} + \gamma\,{}_B\sigma'\;\varepsilon^{nqk}\, v_q/c\,,\\ {}_B\sigma^{*kt} &= \gamma\,{}_E\sigma'\;\varepsilon^{tkn}\, v_n/c + \gamma\,{}_B\sigma'\; V^{2kt}\,.\end{aligned} \qquad (2.21)$$

In the observer's frame the medium, isotropic in the comoving frame, is bianisotropic with v_q as distinguished direction.

3. Ohm's Law for the Total Current

a) Covariant Formulation

The connection of the total four-dimensional current j^ν, instead of the conduction current $(\delta^\nu_\mu + u^\nu\, u_\mu)\, j^\mu$ with the field tensor $F^{\lambda\varkappa}$ leads to an Ohm's Law in the form

$$j^\nu = \tfrac{1}{2}\,\hat{\sigma}^\nu{}_{\lambda\varkappa}\, F^{\lambda\varkappa}\,. \qquad (3.1)$$

Now $\hat{\sigma}^\nu{}_{\lambda\varkappa}$ contains not only the response of the conduction current, but that of the charge transport $-\,u^\nu u_\mu j^\mu$, too. Both currents are connected by the continuity equation

$$\partial_\nu\, j^\nu = 0\,. \qquad (3.2)$$

Equation (3.2) and the skew-symmetry of $F^{\lambda\varkappa}$ yield

$$\partial_\nu\, \hat{\sigma}^\nu{}_{\lambda\varkappa}\, F^{\lambda\varkappa} = 0\,,\quad \hat{\sigma}^\nu{}_{\lambda\varkappa} = -\,\hat{\sigma}^\nu{}_{\varkappa\lambda}\,, \qquad (3.3)$$

which reduce the originally 64 components of $\hat{\sigma}^\nu{}_{\lambda\varkappa}$ to 18.

Like (2.5) one can formulate Ohm's Law with the help of the four-fields E^λ, B^λ as

$$j^\nu = {}_E\hat{\sigma}^\nu{}_\lambda\, E^\lambda + {}_B\hat{\sigma}^\nu{}_\varphi\, c\, B^\varphi\,, \qquad (3.4)$$

which leads to the same representation for $\hat{\sigma}^\nu{}_{\lambda\varkappa}$ by ${}_{E,B}\hat{\sigma}^\nu{}_\lambda$ as (2.7), viz.

$$\hat{\sigma}^\nu{}_{\lambda\varkappa} = {}_E\hat{\sigma}^\nu{}_\lambda\, u_\varkappa - {}_E\hat{\sigma}^\nu{}_\varkappa\, u_\lambda + {}_B\hat{\sigma}^\nu{}_\varphi\, \varepsilon_{\lambda\varkappa}{}^{\varphi\psi}\, u_\psi\,. \qquad (3.5)$$

Unlike the conduction current the total current is not orthogonal to the four-velocity, but satisfies the continuity Equation (3.2). This means, that in the Fourier space the total current is perpendicular to the four-dimensional wave vector

$$k^\lambda := (\omega/c, k^1, k^2, k^3)\,.$$

Therefore analogous considerations as in Chapt. 2 lead now, instead of (2.9), to the following condition and assumption, respectively

$$k_\nu\, {}_{E,B}\hat{\sigma}^\nu{}_\lambda = {}_{E,B}\hat{\hat{\sigma}}\, u_\lambda\,,\quad {}_{E,B}\hat{\sigma}^\nu{}_\lambda\, k^\lambda = {}_{E,B}\hat{\hat{\sigma}}\, u^\nu\,, \qquad (3.6)$$

which hold in the Fourier space and reduce the 16 components of $\hat{\sigma}^\nu{}_{\lambda\varkappa}$ to 10, with ${}_{E,B}\hat{\hat{\sigma}}$ being arbitrary. In the following we will restrict ourselves to the Fourier space.

Equations (3.6) make ${}_{E,B}\hat{\sigma}^\nu{}_\lambda$ in general, for non-dispersive media as well, dependent on the wave vector k_l and frequency ω. The reason is the connection between the charge response and current response by the continuity Equation (3.2).

In order that in the comoving frame Eq. (3.4) becomes for $\nu = l$ the usual Ohm's Law (2.10), the pure space components ${}_{E,B}\hat{\sigma}'^{l}{}_k$ have to be equal to the components of the three-dimensional conductivity tensor ${}_{E,B}\sigma'^{l}{}_{k}$. For $\nu = 0$ Eq. (3.4) expresses the charge density ϱ' through the current response. The other components of ${}_{E,B}\hat{\sigma}'^{\nu}{}_{\lambda}$ in the comoving frame are determined from (3.6) by separation of the space and time components. One obtains

$$
\begin{aligned}
{}_{E,B}\hat{\sigma}'^{l}{}_{k} &= {}_{E,B}\sigma'^{l}{}_{k}\,, \quad {}_{E,B}\hat{\sigma}'^{0}{}_{0} = {}_{E,B}\hat{\hat{\sigma}}\,,\\
{}_{E,B}\hat{\sigma}'^{0}{}_{k} &= \frac{1}{\omega'}\, k'_{l}\, {}_{E,B}\sigma'^{l}{}_{k}\,,\\
{}_{E,B}\hat{\sigma}'^{l}{}_{0} &= \frac{1}{\omega'}\, k'^{k}\, {}_{E,B}\sigma'^{l}{}_{k}\,.
\end{aligned} \tag{3.7}
$$

Contrary to (2.11) the zeroth row and column do not vanish, they represent the charge response.

Since $\hat{\sigma}^{\nu}{}_{\lambda\varkappa}$ has the same structure (3.5) as (2.7) one can apply the same symmetry considerations as in Chapter 2.

The formulation (3.1) of Ohm's Law is certainly less convenient than (2.1), as one has to work from the beginning in the Fourier space in order to avoid derivatives and the material tensors depend always on the wave vector and the frequency.

b) Three-Dimensional Formulation

If one connects, instead of the three-dimensional conduction current, the three-dimensional total current with the field components one gets the Ohm's Law for the total current

$$
j^{k} = {}_{E}\hat{\sigma}^{*k}{}_{n}\, E^{n} + {}_{B}\hat{\sigma}^{*k}{}_{s}\, c\, B^{s}\,, \tag{3.8}
$$

where the asterisks again distinguish the three-dimensional tensors in (3.8) from the space components of the four-dimensional ones in (3.4). In the comoving frame ${}_{E,B}\hat{\sigma}^{*l}{}_{k}$ is equal to ${}_{E,B}\sigma'^{l}{}_{k}$ and (3.8) takes the form (2.10). The same considerations and procedures as in Chapt. 2 lead to the relations between $\hat{\sigma}^{\nu\lambda\varkappa}$ and ${}_{E,B}\hat{\sigma}^{*lk}$ in the form

$$
\hat{\sigma}^{l0n} = -\, {}_{E}\hat{\sigma}^{*ln}\,, \quad \hat{\sigma}^{lmn} = \varepsilon^{smn}\, {}_{B}\hat{\sigma}^{*l}{}_{s} \tag{3.9}
$$

and gives the transformation relations between the three-dimensional material tensors

$$
\begin{aligned}
{}_{E}\hat{\sigma}^{*lk} &= \gamma\left(\Gamma^{l}_{r} - \frac{v^{l}}{\omega'}\, k'_{r}\right)\left({}_{E}\sigma'^{rs}\, V^{k}_{s} + {}_{B}\sigma'^{r}{}_{s}\, \varepsilon^{kqs}\, \frac{v_{q}}{c}\right)\\
{}_{B}\hat{\sigma}^{*lk} &= \gamma\, \varepsilon^{mnk}\left(\Gamma^{l}_{r} - \frac{v^{l}}{\omega'}\, k'_{r}\right)\\
&\quad\cdot\left(\frac{v_{n}}{c}\, {}_{E}\sigma'^{r}{}_{m} - \frac{v_{m}}{c}\, {}_{E}\sigma'^{r}{}_{n} + {}_{B}\sigma'^{r}{}_{s}\, V^{qs}\, \varepsilon_{mnq}\right)
\end{aligned} \tag{3.10}
$$

with $V^{l}_{r} = \Gamma^{-1l}_{\ \ r}$ and Γ^{l}_{r} given by (2.13b). The terms proportional to Γ^{l}_{r} represent in the observer's frame the conduction current of the comoving frame, while the other terms represent the convection current of the comoving frame in the observer's frame.

4. The Polarization Model

In the comoving frame the response of the free charges is sometimes expressed by the response of the bound charges in the so-called polarization model using the relation [10]

$$
j'^{l} = \frac{\partial}{\partial t'}\, P'^{l} + \varepsilon^{lkm}\, \partial'_{k}\, M'_{m}\,, \quad \varrho' = -\, \partial'_{k}\, P'^{k}\,. \tag{4.1}
$$

For the electric and magnetic polarizations P'^{k}, M'^{k} one formulates the constitutive equations as

$$
\begin{aligned}
P'^{l}/\varepsilon_{0} &= {}_{E}\chi'^{l}{}_{k}\, E'^{k} + {}_{B}\chi'^{l}{}_{k}\, c\, B'^{k}\,,\\
\mu_{0}\, c\, M'^{l} &= {}_{E}\lambda'^{l}{}_{k}\, E'^{k} + {}_{B}\lambda'^{l}{}_{k}\, c\, B'^{k}\,,
\end{aligned} \tag{4.2}
$$

where ${}_{E}\chi'^{l}{}_{k}({}_{B}\chi'^{l}{}_{k})$ and ${}_{E}\lambda'^{l}{}_{k}({}_{B}\lambda'^{l}{}_{k})$, respectively, represent the response of the electric and magnetic dipoles, respectively, to the electric field (magnetic flux).

Equations (4.1) together with the constitutive Eqs. (4.2), (2.10) enables to express the electric and magnetic susceptibility tensors ${}_{E,B}\chi'^{l}{}_{k}$, ${}_{E,B}\lambda'^{l}{}_{k}$ through the conductivity tensors ${}_{E,B}\sigma'^{l}{}_{k}$. Since in the three equations of (4.1) for j'^{l} the six components of the polarizations P'^{l}, M'^{l} are not uniquely determined by the three components of the current vector j'^{l}, the susceptibility tensors are not uniquely determined either. This under-determination leads to a great number of relations between the susceptibility and conductivity tensors; a very common one among them is

$$
{}_{E,B}\sigma'^{l}{}_{k}/\varepsilon_{0} = -\, i\, \omega'\, {}_{E,B}\chi'^{l}{}_{k}\,, \quad {}_{E,B}\lambda'^{l}{}_{k} = 0\,, \tag{4.3}
$$

which corresponds to the combination of the dielectric and conductivity tensor to the so-called effective dielectric tensor

$$
\varepsilon'_{\text{eff}}{}^{l}{}_{k} := \varepsilon_{0}\left(\delta^{l}_{k} + {}_{E}\chi'^{l}{}_{k}\right) + \frac{i}{\omega'}\, {}_{E}\sigma'^{l}{}_{k}\,. \tag{4.4}
$$

a) Covariant Formulation of the Constitutive Relations

The electric and magnetic polarizations can be combined to a skew-symmetric polarization tensor $M^{\nu\mu}$ [5] in the form

$$
M^{0l} := c\, P^{l} = -\, M^{l0}\,, \quad M^{lk} := \varepsilon^{lkm}\, M_{m}\,, \tag{4.5}
$$

which allows to formulate (4.1) as

$$\partial_\nu M^{\nu\mu} = j^\mu . \tag{4.6}$$

Analogous to the excitation tensor in [2], [3] the polarization tensor $M^{\nu\mu}$ is connected with the field tensor $F^{\lambda\varkappa}$ by a tensor of order four:

$$M^{\nu\mu} = \tfrac{1}{2}(\varepsilon_0/\mu_0)^{1/2}\, \chi^{\nu\mu}{}_{\lambda\varkappa}\, F^{\lambda\varkappa} . \tag{4.7}$$

The skew-symmetry of both $M^{\nu\mu}$ and $F^{\lambda\varkappa}$ leads to the following symmetry relations of the susceptibility tensor $\chi^{\nu\mu}{}_{\lambda\varkappa}$:

$$\chi^{\nu\mu}{}_{\lambda\varkappa} = -\chi^{\mu\nu}{}_{\lambda\varkappa} = -\chi^{\nu\mu}{}_{\varkappa\lambda} = \chi^{\mu\nu}{}_{\varkappa\lambda} , \tag{4.8}$$

which reduce the originally 256 components to 36.

The four-fields E^λ, B^λ (2.3a) and the four-polarizations P^λ, M^λ defined by

$$c\,P^\lambda := M^{\lambda\varkappa} u_\varkappa , \quad M^\lambda := \mathscr{M}^{\lambda\varkappa} u_\varkappa , \tag{4.9}$$

$$c\,P^\lambda = (P^l v_l , c\,P^l + \varepsilon^{lkm} v_k M_m/c)/(1 - v^2/c^2)^{1/2} ,$$

$$M^\lambda = (M^l v_l/c , M^l - \varepsilon^{lkm} v_k P_m)/(1 - v^2/c^2)^{1/2} ,$$

with

$$\mathscr{M}^{\lambda\varkappa} := \tfrac{1}{2}\,\varepsilon^{\lambda\varkappa\varphi\psi} M_{\varphi\psi}$$

allow to formulate constitutive relations in the form

$$\begin{aligned} P^\nu/\varepsilon_0 &= {}_E\chi^\nu{}_\lambda E^\lambda + {}_B\chi^\nu{}_\lambda\, c\, B^\lambda , \\ \mu_0\, c\, M^\nu &= {}_E\lambda^\nu{}_\lambda E^\lambda + {}_B\lambda^\nu{}_\lambda\, c\, B^\lambda . \end{aligned} \tag{4.10}$$

Like E^λ, B^λ (2.3b) the four-polarizations P^λ, M^λ are perpendicular to the four-velocity:

$$P^\lambda u_\lambda = 0 , \quad M^\lambda u_\lambda = 0 . \tag{4.11}$$

The orthogonality relations (2.3b), (4.11) lead under the same considerations, as in Chapt. 2, to the following conditions and assumptions, respectively, for the four-dimensional material tensors of order two:

$$\begin{aligned} u_\nu\, {}_{E,B}\chi^\nu{}_\lambda &= {}_{E,B}\bar{\chi}\, u_\lambda , \quad {}_{E,B}\chi^\nu{}_\lambda u^\lambda = {}_{E,B}\bar{\chi}\, u^\nu , \\ u_\nu\, {}_{E,B}\lambda^\nu{}_\lambda &= {}_{E,B}\bar{\lambda}\, u_\lambda , \quad {}_{E,B}\lambda^\nu{}_\lambda u^\lambda = {}_{E,B}\bar{\lambda}\, u^\nu \end{aligned} \tag{4.12}$$

which again reduce the 16 components of each of the four tensors to 10, with ${}_{E,B}\bar{\chi}$, ${}_{E,B}\bar{\lambda}$ being arbitrary.

In the comoving frame Eqs. (4.10) become for $\nu = 0$ the identities $0 = 0$. In order that for $\nu = l$ Eqs. (4.10) become the usual three-dimensional constitutive relations (4.2), the space components of the four-dimensional tensors ${}_{E,B}\chi'^\nu{}_\lambda$, ${}_{E,B}\lambda'^\nu{}_\lambda$ have to be equal to the components of the three-dimensional susceptibility tensors ${}_{E,B}\chi'^l{}_k$, ${}_{E,B}\lambda'^l{}_k$. The remaining components can be determined by (4.12). Thus one obtains as the explicit representation of the four-dimensional susceptibility tensors of order two in the comoving frame:

$$\begin{aligned} &{}_{E,B}\chi'^l{}_k = {}_{E,B}\chi'^l{}_k , \quad {}_{E,B}\chi'^l{}_0 = 0 = {}_{E,B}\chi'^0{}_k , \\ &{}_{E,B}\chi'^0{}_0 = {}_{E,B}\bar{\chi} , \\ &{}_{E,B}\lambda'^l{}_k = {}_{E,B}\lambda'^l{}_k , \quad {}_{E,B}\lambda'^l{}_0 = 0 = {}_{E,B}\lambda'^0{}_k , \\ &{}_{E,B}\lambda'^0{}_0 = {}_{E,B}\bar{\lambda} . \end{aligned} \tag{4.13}$$

The representations in the observer's frame can be calculated by Lorentz transformations according to (2.13).

Analogously to (2.4) one can represent $M^{\nu\mu}$ by P^ν, M^μ as

$$M^{\nu\mu} = c\, u^\nu P^\mu - c\, u^\mu P^\nu - \varepsilon^{\nu\mu\varphi\psi} u_\varphi M_\psi . \tag{4.14}$$

Inserting (4.10) into the above equation, one gets with the definition of E^λ, B^λ (2.3a) an equation connecting the polarization with the field tensor $F^{\lambda\varkappa}$. A comparison with (4.7) yields after some rearrangements the representation of $\chi^{\nu\mu}{}_{\lambda\varkappa}$:

$$\begin{aligned} \chi^{\nu\mu}{}_{\lambda\varkappa} = &- ({}_E\chi^\nu{}_\lambda u_\varkappa - {}_E\chi^\nu{}_\varkappa u_\lambda)\, u^\mu \\ &+ ({}_E\chi^\mu{}_\lambda u_\varkappa - {}_E\chi^\mu{}_\varkappa u_\lambda)\, u^\nu \\ &+ {}_B\lambda_\sigma{}^\varrho\, \varepsilon^{\nu\mu\varphi\sigma}\, \varepsilon_{\lambda\varkappa\psi\varrho}\, u_\varphi u^\psi \\ &- ({}_B\chi^\mu{}_\varrho u^\nu - {}_B\chi^\nu{}_\varrho u^\mu)\, \varepsilon_{\lambda\varkappa}{}^{\psi\varrho} u_\psi \\ &- ({}_E\lambda^\sigma{}_\lambda u_\varkappa - {}_E\lambda^\sigma{}_\varkappa u_\lambda)\, \varepsilon^{\nu\mu}{}_{\varphi\sigma} u^\varphi , \end{aligned} \tag{4.15}$$

where the 36 components of $\chi^{\nu\mu}{}_{\lambda\varkappa}$ are uniquely determined by the components of each of the four susceptibility tensors ${}_{E,B}\chi^{\nu\mu}$, ${}_{E,B}\lambda^{\nu\mu}$. The representation (4.15) is the most general form by which a tensor of order four, connecting two skew-symmetric tensors of order two, can be represented by tensors of order two. It includes as a special case the expressions found by Marx [2] and Schmutzer [3] for bound charges in media anisotropic in the comoving frame (${}_B\chi'^{\nu\mu} = {}_E\lambda'^{\nu\mu} = 0$) and the general relativistic expressions found by Maugin [11] for media isotropic in the comoving frame.

If we use the constitutive relation (4.7) for free charges in the polarization model, one can fix arbitrarily some of the material tensors (4.15) because of the underdetermination of the susceptibility tensors, which has been discussed earlier. If we generalize the convention (4.3), viz. ${}_E\lambda^{\nu\mu} = {}_B\lambda^{\nu\mu} = 0$, Eq. (4.15) becomes

$$\begin{aligned} \chi^{\nu\mu}{}_{\lambda\varkappa} = &({}_E\chi^\mu{}_\lambda u_\varkappa - {}_E\chi^\mu{}_\varkappa u_\lambda)\, u^\nu \\ &- ({}_E\chi^\nu{}_\lambda u_\varkappa - {}_E\chi^\nu{}_\varkappa u_\lambda)\, u^\mu \\ &- ({}_B\chi^\mu{}_\varrho u^\nu - {}_B\chi^\nu{}_\varrho u^\mu)\, \varepsilon_{\lambda\varkappa}{}^{\psi\varrho} u_\psi . \end{aligned} \tag{4.16}$$

This is considerably more complicated than the expression (2.7) for the conductivity tensor.

Besides of (4.8) one often imposes additional symmetry relations, for example [9], [4],

$$\chi^{\nu\mu}{}_{\lambda\varkappa} = \chi_{\lambda\varkappa}{}^{\nu\mu} \tag{4.17}$$

which reduces the 36 components to 21 and requires certain symmetry properties of the material tensors of order two. Insertion of (4.15) into (4.17) yields the following symmetry properties of ${}_{E,B}\chi^{\nu}{}_{\mu}$, ${}_{E,B}\lambda^{\nu}{}_{\mu}$:

$${}_E\chi^{\nu}{}_{\mu} = {}_E\chi_{\mu}{}^{\nu}, \quad {}_B\lambda^{\nu}{}_{\mu} = {}_B\lambda_{\mu}{}^{\nu}, \quad {}_B\chi^{\nu}{}_{\mu} = {}_E\lambda_{\mu}{}^{\nu}, \tag{4.18}$$

which have to be satisfied in order that (4.17) holds. As the symmetry relations are Lorentz invariant relations, (4.18) can easily be transferred to the three-dimensional tensors in the comoving frame. One recognizes that for media anisotropic in the comoving frame (${}_E\chi'^{\nu}{}_{\mu} = {}_B\lambda'^{\nu}{}_{\mu} = 0$) relations (4.18) require the symmetry of the material tensors, which is often satisfied by dielectric media, but not for plasmas.

The additional symmetry relation [9]

$$\chi^{[\nu\mu\lambda\varkappa]} = 0 \tag{4.19a}$$

reduces the components of $\chi^{\nu\mu\lambda\varkappa}$ to 20. Eqs. (4.8) and (4.17) yield for (4.19a)

$$\chi^{\nu\mu\lambda\varkappa} + \chi^{\nu\varkappa\mu\lambda} + \chi^{\nu\lambda\varkappa\mu} = 0\,. \tag{4.19b}$$

Insertion of (4.15) into (4.19b) leads in the comoving frame, according to (2.16b), to the vanishing of the traces ${}_B\chi'^{l}{}_{l}$, ${}_E\lambda'^{l}{}_{l}$, and thus, with (4.18)

$${}_B\chi^{\nu}{}_{\nu} = {}_E\lambda_{\nu}{}^{\nu} = {}_B\overline{\chi} = {}_E\overline{\lambda}\,. \tag{4.20}$$

This does not impose an additional condition on media anisotropic in the comoving frame, but on bianisotropic media.

b) Three-Dimensional Formulation

In three-dimensional form the constitutive relations (4.10) are:

$$\begin{aligned} P^l/\varepsilon_0 &= {}_E\chi^{*l}{}_k\, E^k + {}_B\chi^{*l}{}_k\, c\, B^k\,, \\ \mu_0\, c\, M^l &= {}_E\lambda^{*l}{}_k\, E^k + {}_B\lambda^{*l}{}_k\, c\, B^k\,, \end{aligned} \tag{4.21}$$

where again the asterisks distinguish the three-dimensional tensors in (4.21) from the space components of the four-dimensional ones in (4.10). In the comoving frame (4.21) coincides with (4.2).

If one splits in (4.7) the polarization tensor on the left side into $M^{0k} = c\,P^k$ and $M^{lk} = \varepsilon^{lkr}\, M_r$ (4.5), respectively, and on the right hand side the summation over the greek indices in a summation over 0 and l, one gets by comparison with (4.21), where E_k and $c\,B_k$ have been substituted by the expressions (2.18), the following relations between the components of the susceptibility tensor of order four and the three-dimensional tensors of order two:

$$\begin{aligned} \chi^{0k0n} &= -\,{}_E\chi^{*kn}\,, & \chi^{0kmn} &= \varepsilon^{smn}\,{}_B\chi^{*k}{}_s\,, \\ \chi^{lk0n} &= -\,\varepsilon^{lkr}\,{}_E\lambda^{*}{}_r{}^n\,, & \chi^{lkmn} &= \varepsilon^{lkr}\,\varepsilon^{smn}\,{}_B\lambda^{*}{}_{rs}\,. \end{aligned} \tag{4.22}$$

The representation of $\chi^{\nu\mu\lambda\varkappa}$ in the observer's frame through the components of the three-dimensional tensors ${}_{E,B}\chi'^{l}{}_k$, ${}_{E,B}\lambda'^{l}{}_k$ in the comoving frame is given by a Lorentz transformation (according to (2.13)) of (4.22) taken in the comoving frame. The comparison of the resulting expressions with (4.22) in the observer's frame, leads after some manipulations to the transformation relations of the three-dimensional material tensors:

$$\begin{aligned} \frac{1}{\gamma^2}\,{}_E\chi^{*lk} = {} & V^{lm}\,V^{nk}\,{}_E\chi'_{mn} - V^{lm}\,\varepsilon^{kns}\,\beta_s\,{}_B\chi'_{mn} \\ & - V^{nk}\,\varepsilon^{lsm}\,\beta_s\,{}_E\lambda'_{mn} \\ & + \varepsilon^{lsm}\,\varepsilon^{knr}\,\beta_s\,\beta_r\,{}_B\lambda'_{mn}\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \frac{1}{\gamma^2}\,{}_B\chi^{*lk} = {} & V^{lm}\,\varepsilon^{kns}\,\beta_s\,{}_E\chi'_{mn} - V^{lm}\,V^{nk}\,{}_B\chi'_{mn} \\ & - \varepsilon^{lsm}\,\varepsilon^{knr}\,\beta_s\,\beta_r\,{}_E\lambda'_{mn} \\ & - V^{nk}\,\varepsilon^{lsm}\,\beta_s\,{}_B\lambda'_{mn} \end{aligned}$$

$$\begin{aligned} \frac{1}{\gamma^2}\,{}_E\lambda^{*lk} = {} & V^{nk}\,\varepsilon^{lsm}\,\beta_s\,{}_E\chi'_{mn} \\ & - \varepsilon^{lsm}\,\varepsilon^{knr}\,\beta_s\,\beta_r\,{}_B\chi'_{mn} \\ & + V^{lm}\,V^{nk}\,{}_E\lambda'_{mn} \\ & - V^{lm}\,\varepsilon^{knr}\,\beta_r\,{}_B\lambda'_{mn}\,, \end{aligned} \tag{4.23}$$

$$\begin{aligned} \frac{1}{\gamma^2}\,{}_B\lambda^{*lk} = {} & \varepsilon^{lsm}\,\varepsilon^{knr}\,\beta_s\,\beta_r\,{}_E\chi'_{mn} \\ & + V^{nk}\,\varepsilon^{lsm}\,\beta_s\,{}_B\chi'_{mn} \\ & + V^{lm}\,\varepsilon^{knr}\,\beta_r\,{}_E\lambda'_{mn} \\ & + V^{lm}\,V^{nk}\,{}_B\lambda'_{mn} \end{aligned}$$

with $V^{lm} = \Gamma^{-1lm}$ (2.20) and $\beta^l = v^l/c$.

Concluding Remarks

The three different forms of constitutive relations for moving plasmas — Ohm's Law, Ohm's Law for the total current, and the constitutive relation in the polarization model —, which we have formulated in four and three dimensions, equivalently describe the properties of a plasma. Ohm's Law has proved to be most convenient, because Ohm's Law for the total current is restricted to the Fourier space and makes the material tensors in general, for non-dispersive media as well, dependent on the wave vector and the frequency. The constitutive relations in the polarization model lead to much longer expressions.

As the constitutive relations in the polarization model serve primarily to describe properties of bound charges, the results for constitutive relations for bound charges by Marx [2], Schmutzer [3], [6] and Kong [4] are recovered by our results and generalized to bianisotropic media, respectively.

Although we had restricted ourselves to plasmas moving with constant velocity, the constitutive relations and the properties of the material tensors, which are formulated covariantly, are valid generally for arbitrary moving media.

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